SHARPENING AND GENERALIZATIONS OF SHAFER-FINK'S DOUBLE INEQUALITY FOR THE ARC SINE FUNCTION

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Abstract. In this paper, we sharpen and generalize Shafer-Fink's double inequality for the arc sine function.

1. Introduction and main results

In [3, p. 247, 3.4.31], it was listed that the inequality

$$\arcsin x > \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}} \tag{1}$$

holds for $0 < x < 1$. It was also pointed out in [3, p. 247, 3.4.31] that these inequalities are due to R. E. Shafer, but no a related reference is cited. By now we do not know the very original source of inequalities in (1).

In the first part of the short paper [1], the inequality between the very ends of (1) was recovered and an upper bound for the arc sine function was also established as follows:

$$\frac{3x}{2+\sqrt{1-x^2}} \le \arcsin x \le \frac{\pi x}{2+\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{2}$$

Therefore, we call (2) the Shafer-Fink's double inequality for the arc sine function.

In [2], the right-hand side inequality in (2) was improved to

$$\arcsin x \le \frac{\pi x/(\pi-2)}{2/(\pi-2)+\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{3}$$

In [13], the inequality (3) was recovered and the following Shafer-Fink type inequalities were derived:

$$\frac{\pi(4-\pi)x}{2/(\pi-2)+\sqrt{1-x^2}} \le \arcsin x, \quad 0 \le x \le 1; \tag{4}$$

$$\frac{(\pi/2)x}{1+\sqrt{1-x^2}} \le \arcsin x, \quad 0 \le x \le 1. \tag{5}$$

Note that the lower bounds in (2), (4) and (5) are not included each other.

The main aim of this paper is to sharpen and generalize the above Shafer-Fink type double inequalities.

Our main results can be stated as follows.

Theorem 1. *For $\alpha \in \mathbb{R}$ and $x \in (0,1]$, the function*

$$f_\alpha(x) = \left(\alpha + \sqrt{1-x^2}\right)\frac{\arcsin x}{x} \tag{6}$$

is strictly

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(1) *increasing if and only if* $\alpha \ge 2$*;*
(2) *decreasing if and only if* $\alpha \le \frac{\pi}{2}$.

Moreover, when $\frac{\pi}{2} < \alpha < 2$*, the function* $f_\alpha(x)$ *has a unique minimum on* $(0,1)$.

As straightforward consequences of Theorem 1, the following double inequalities may be derived readily.

Theorem 2. *If* $\alpha \ge 2$*, the double inequality*

$$\frac{(\alpha+1)x}{\alpha+\sqrt{1-x^2}} \le \arcsin x \le \frac{(\pi\alpha/2)x}{\alpha+\sqrt{1-x^2}} \tag{7}$$

holds on $[0,1]$*. If* $0 < \alpha \le \frac{\pi}{2}$*, the inequality* (7) *reverses. If* $\frac{\pi}{2} < \alpha < 2$*, then the inequality*

$$\frac{4\bigl(1-1/\alpha^2\bigr)x}{\alpha+\sqrt{1-x^2}} \le \arcsin x \le \frac{\max\{\pi\alpha/2, \alpha+1\}x}{\alpha+\sqrt{1-x^2}} \tag{8}$$

holds on $[0,1]$.

2. Remarks

Before proving our theorems, we would like to give several remarks on them.

Remark 1. Letting $x = \sin t$ for $t \in \bigl[0, \frac{\pi}{2}\bigr]$ yields the restatement of Theorem 2 as follows:

(1) If $\alpha \ge 2$, then

$$\frac{(\alpha+1)\sin t}{\alpha+\cos t} \le t \le \frac{(\pi\alpha/2)\sin t}{\alpha+\cos t}, \quad 0 \le t \le \frac{\pi}{2}. \tag{9}$$

(2) If $0 < \alpha \le \frac{\pi}{2}$, the inequality (7) reverses.
(3) If $\frac{\pi}{2} < \alpha < 2$, then

$$\frac{4\bigl(1-1/\alpha^2\bigr)\sin t}{\alpha+\cos t} \le t \le \frac{\max\{\pi\alpha/2, \alpha+1\}\sin t}{\alpha+\cos t}, \quad 0 \le t \le \frac{\pi}{2}. \tag{10}$$

For more information on the inequalities in (9) and (10), please refer to [5] and closely-related references therein.

Remark 2. The Shafer-Fink's double inequality (2) is the special case $\alpha = 2$ in (7).

Remark 3. Taking $\alpha = \frac{\pi}{2}$ in (7) gives

$$\frac{(\pi^2/4)x}{\pi/2+\sqrt{1-x^2}} \le \arcsin x \le \frac{(\pi/2+1)x}{\pi/2+\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{11}$$

This improves the inequality (5) and recovers the right-hand side inequality of Theorem 8 on [13, p. 61].

The left-hand side inequalities in (4) and (11) are not included each other.

The lower bound in (11) and those in (1) are not included each other.

Remark 4. Since $\frac{\pi\alpha}{2} = \alpha+1$ has a unique root $\alpha = \frac{2}{\pi-2} \in \bigl(\frac{\pi}{2}, 2\bigr)$, the inequality (3) follows from taking $\alpha = \frac{2}{\pi-2}$ in (8).

Remark 5. Let

$$h_x(\alpha) = \frac{1-1/\alpha^2}{\alpha+\sqrt{1-x^2}}$$

for $\frac{\pi}{2} < \alpha < 2$ and $x \in (0,1)$. Then

$$\alpha\bigl(3-\alpha^2\bigr) < \alpha^3\bigl(\alpha+\sqrt{1-x^2}\,\bigr)^2 h_x'(\alpha) = 3\alpha - \alpha^3 + 2\sqrt{1-x^2} < 2 + 3\alpha - \alpha^3.$$

This means that

(1) when $\frac{\pi}{2} < \alpha \le \sqrt{3}$ the function $\alpha \mapsto h_x(\alpha)$ is increasing;

(2) when $\sqrt{3} < \alpha < 2$ the function $\alpha \mapsto h_x(\alpha)$ attains its maximum

$$\frac{4\cos^2\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right]-1}{4\left\{2\cos\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right]+\sqrt{1-x^2}\right\}\cos^2\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right]}$$

at the point

$$2\cos\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right], \quad x \in (0,1).$$

Therefore, the following two sharp inequalities may be derived from the left-hand side inequality in (8) for $x \in (0,1)$:

$$\arcsin x > \frac{(8/3)x}{\sqrt{3}+\sqrt{1-x^2}}, \tag{12}$$

$$\arcsin x > \frac{x\left\{4\cos^2\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right]-1\right\}}{\left\{2\cos\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right]+\sqrt{1-x^2}\right\}\cos^2\left[\frac{1}{3}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\right]}. \tag{13}$$

By the famous software MATHEMATICA 7.0, we reveal that the inequality (13) is better than the left-hand side inequality in (2) and the inequalities (4) and (12) and that it does not include the inequality (5) and the left-hand side inequality (11).

Remark 6. The method to prove Theorem 1 and Theorem 2 in next section have been used in [4, 5, 6, 7, 8, 9, 10, 11, 12] and closely-related references therein.

Remark 7. The method used in next section to prove Theorem 1 and Theorem 2 is more elementary than the one utilized in [1, 2, 13, 14, 15].

Remark 8. This paper is a slightly modified version of the preprint [9].

3. Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1. Direct differentiation yields

$$\begin{aligned} f_\alpha'(x) &= \frac{\alpha+1/\sqrt{1-x^2}}{x^2}\left[\frac{x\left(\alpha+\sqrt{1-x^2}\right)}{1+\alpha\sqrt{1-x^2}}-\arcsin x\right] \\ &\triangleq \frac{\alpha+1/\sqrt{1-x^2}}{x^2}h_\alpha(x), \\ h_\alpha'(x) &= \frac{x^2\left(\alpha^2-2-\alpha\sqrt{1-x^2}\right)}{\left(1+\alpha\sqrt{1-x^2}\right)^2\sqrt{1-x^2}}. \end{aligned}$$

Because

$$\alpha^2-\alpha-2 \le \alpha^2-2-\alpha\sqrt{1-x^2} \le \alpha^2-2$$

on $[0,1]$, the derivative $h_\alpha'(x)$ is negative (or positive respectively) when $0<\alpha\le\sqrt{2}$ (or $\alpha \ge 2$ respectively). Moreover, if $\sqrt{2} < \alpha < 2$, the derivative $h_\alpha'(x)$ has a unique zero on $(0,1)$. As a result, the function $h_\alpha(x)$ is increasing (or decreasing respectively) when $\alpha \ge 2$ (or $0 < \alpha \le \sqrt{2}$ respectively) and has a unique minimum on $(0,1)$ when $\sqrt{2} < \alpha < 2$. It is easy to obtain that $h_\alpha(0)=0$ and $h_\alpha(1)=\alpha-\frac{\pi}{2}$. Hence,

(1) when $\alpha \ge 2$, the function $h_\alpha(x)$ and $f_\alpha'(x)$ are positive, and so $f_\alpha(x)$ is strictly increasing on $(0,1)$;
(2) when $0<\alpha\le\sqrt{2}$, the function $h_\alpha(x)$ and $f_\alpha'(x)$ are negative, and so $f_\alpha(x)$ is strictly decreasing on $(0,1)$;
(3) when $\sqrt{2} < \alpha < 2$ and $\alpha \le \frac{\pi}{2}$, the function $h_\alpha(x)$ and $f_\alpha'(x)$ are also negative, and so $f_\alpha(x)$ is also strictly decreasing on $(0,1)$;

(4) when $\sqrt{2} < \alpha < 2$ and $\alpha > \frac{\pi}{2}$, the function $h_\alpha(x)$ and $f'_\alpha(x)$ have the same unique zero on $(0,1)$, and so the function $f_\alpha(x)$ has a unique minimum on $(0,1)$.

On other hand, the derivative $f'_\alpha(x)$ may be rearranged as

$$\begin{aligned} f'_\alpha(x) &= \frac{1}{x^2}\left[x\left(1+\frac{\alpha}{\sqrt{1-x^2}}\right)-\left(\alpha+\frac{1}{\sqrt{1-x^2}}\right)\arcsin x\right] \\ &\triangleq \frac{1}{x^2}H_\alpha(x), \\ H'_\alpha(x) &= -\frac{x\bigl[x\bigl(\sqrt{1-x^2}-\alpha\bigr)+\arcsin x\bigr]}{(1-x^2)^{3/2}}. \end{aligned}$$

When $\alpha \le 0$, the derivative $H'_\alpha(x)$ is negative, and so the function $H_\alpha(x)$ is strictly decreasing on $(0,1)$. From

$$\lim_{x\to 0^+} H_\alpha(x) = 0,$$

it follows that $H_\alpha(x) < 0$ on $(0,1)$. Therefore, when $\alpha \le 0$, the derivative $f'_\alpha(x)$ is negative and the function $f_\alpha(x)$ is strictly decreasing on $(0,1)$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. It is easy to obtain that

$$\lim_{x\to 0^+} f_\alpha(x) = \alpha + 1 \quad \text{and} \quad f_\alpha(1) = \frac{\pi}{2}\alpha.$$

From the monotonicity obtained in Theorem 1, it follows that

(1) when $\alpha \ge 2$, we have

$$\alpha + 1 < \left(\alpha + \sqrt{1-x^2}\right)\frac{\arcsin x}{x} \le \frac{\pi}{2}\alpha \tag{14}$$

on $(0,1]$, which can be rewritten as the inequality (7);

(2) when $0 < \alpha \le \frac{\pi}{2}$, the inequality (14) is reversed;

(3) when $\frac{\pi}{2} < \alpha < 2$, we have

$$\left(\alpha + \sqrt{1-x^2}\right)\frac{\arcsin x}{x} \le \max\left\{\frac{\pi}{2}\alpha, \alpha+1\right\}$$

which can be rearranged as the right-hand side inequality in (8).

On the other hand, when $\frac{\pi}{2} < \alpha < 2$, the minimum point $x_0 \in [0,1]$ of $f_\alpha(x)$ satisfies

$$\frac{\arcsin x_0}{x_0} = \frac{\alpha + \sqrt{1-x_0^2}}{1+\alpha\sqrt{1-x_0^2}}.$$

Hence, the minimum equals

$$f_\alpha(x_0) = \frac{\bigl(\alpha+\sqrt{1-x_0^2}\,\bigr)^2}{1+\alpha\sqrt{1-x_0^2}} = \frac{(\alpha+u_0)^2}{1+\alpha u_0} \ge 4\left(1-\frac{1}{\alpha^2}\right), \quad u_0 \in [0,1].$$

The right-hand side inequality in (8) follows. The proof of Theorem 2 is proved. □

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